Directed Percolation Phenomena in Asynchronous Elementary Cellular Automata

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Abstract. Cellular automata are discrete dynamical systems that are widely used to model natural systems. Classically they are run with perfect synchrony ; *i.e.*, the local rule is applied to each cell at each time step. A possible modification of the updating scheme consists in applying the rule with a fixed probability, called the synchrony rate. It has been shown in a previous work that varying the synchrony rate continuously could produce a discontinuity in the behaviour of the cellular automaton. This works aims at investigating the nature of this change of behaviour using intensive numerical simulations. We apply a two-step protocol to show that the phenomenon is a phase transition whose critical exponents are in good agreement with the predicted values of directed percolation.

1 Introduction

The research described in this article is motivated by the need to address a general question raised in the modelling activity: "Does a given model keep its behaviour when it is submitted to a perturbation of its updating scheme ?" Of course, this question is too wide to be tackled in all its generality and we choose here to study it in the more narrow context of cellular automata, taking asynchronism as a means of perturbation.

In its classical paradigm, a cellular automaton consists of a collection of finite state automata arranged on a regular grid, which update their state at each time step according to a local rule. Using this formalism, we obtain discrete dynamical systems that are used for modelling spatially extended phenomena governed by a local rule. Such phenomena are to be found in various fields such as physics (*e.g.*, atoms interaction in a crystal), chemistry (*e.g.*, non-stirred reaction-diffusion), biology (*e.g.*, virus spreading), etc. (see [1], chap. 1 for a review). The method used for assessing the validity of a model generally consists in comparing the output produced by the model to some experimental data. We claim that this step is of course necessary but that it is not sufficient: one may also need to examine to which extent the behaviour observed is due to the implicit hypotheses of the model, namely: discretisation of state, regularity of the grid, perfect synchrony of the transitions.

The latter problem was at first addressed in [14] by means of simulation, the evaluation of the change in behaviour remaining qualitative. Other experimental

S. El Yacoubi, B. Chopard, and S. Bandini (Eds.): ACRI 2006, LNCS 4173, pp. 667–675, 2006.

works such as [3,17,15] followed, showing that the update scheme was indeed a key point to study. On the theoretical side, very few results have been obtained so far: the independence on the "update history" was shown undecidable in [9], existence of stationary distributions was studied in [12] and a first classification based on the convergence time was proposed in [7] and extended in [8]. In the work [6], we experimentally showed that the perturbation of the updating scheme of elementary cellular automata may alter significantly the behaviour of some rules while other rules remained robust. We used one of the simplest means of introducing asynchronism in the dynamics: instead of applying the rule simultaneously to all the cells, each cell has a given probability α, called the *synchrony rate* to apply the rule.

This study showed that, among other phenomena, for seven elementary cellular automata, there exists a particular value of the synchrony rate α_c for which a small change of value produces an abrupt change of behaviour. It was then conjectured that this brutal variation could be explained by the existence of a phase transition, more precisely that the universality class[1] of the phase transition was *directed percolation* (DP). We wish to emphasize that this hypothesis was mainly supported by the observation of the space-time diagrams patterns produced near criticality. Previous identification of directed percolation was obtained in other contexts such that probabilistic cellular automata [5] or synchronisation of two copies of cellular automata [10,16]. To our knowledge, the only example of directed percolation induced by asynchronism was given by Blok and Bergersen for the famous Game of Life [4].

2 Description of the Model

Let a ring of n cells be indexed by $\mathcal{L} = \mathbb{Z}/n\mathbb{Z}$, a *configuration* is word on $\{0,1\}^{\mathcal{L}}$. The *density* of a configuration x is the ratio of cells in state 1. An *elementary cellular automaton* (ECA) is described by a function $f : \{0,1\}^3 \to \{0,1\}$ called the *local rule*. Each ECA is indexed according to the usual notation [18].

Using the stochastic asynchronous updating scheme, the local rule f allows to define a probabilistic global rule F which operates on the random variables x^t according to $x^0 = x$ with probability 1 and $x^{t+1} = F(x^t)$ such that:

$$\forall i \in \mathcal{L}, x_i^{t+1} = \begin{cases} f(x_{i-1}^t, x_i^t, x_{i+1}^t) & \text{with probability } \alpha \\ x_i^t & \text{with probability } 1-\alpha \end{cases}$$

By taking $\alpha = 1$, we fall back on the classical synchronous case and as α is decreased, the updtae rule becomes asynchronous while the effect of an update remains unchanged.

The rules that were experimentally detected as showing a brutal change of behaviour for a non-trivial value of α are ECA 6,18,26,50,58,106,146 (only "minimal representative rules" are considered). Figure 1 shows how the variation of

[1] *Universality class* is a term from statistical physics that describes all the different phenomena that obey the same laws near criticality (*e.g.*, see [11] for a review on the directed percolation universality class).

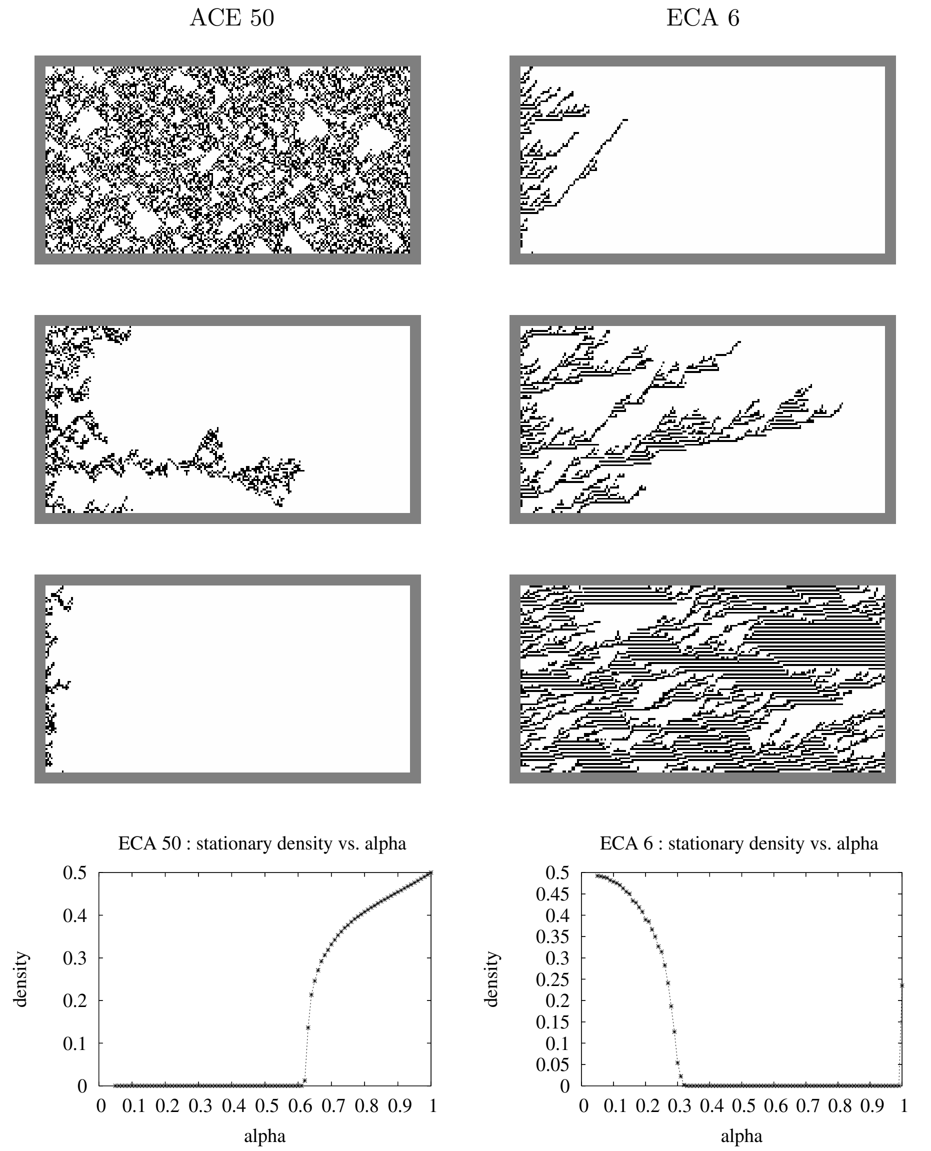

Fig. 1. Two types of phase transitions observed with ECA. **(a)** The upper three lines show the space time diagrams for ECA 50 (left) and ECA 6 (right). Synchrony rate is decreased, from $\alpha = 0.75$ (up) to $\alpha = 0.25$ (down). Time goes from left to right; the time factor is rescaled by a factor $1/\alpha$ (i.e., for $\alpha = 0.25$ only time steps that are multiples of 4 are displayed). **(b)** Lower line : asymptotic density as a function of the synchrony rate ; it is estimated by the computation of the average density obtained on a sampling time $S = 10^3$, after a transient time $T = 10^5$ has elapsed.

synchrony rate affects the behaviour of three such rules. We see that two different behaviours are exhibited:

(a) The system quickly converges to a frozen fixed-point configuration where all the cells are in state 0, we say that we are in the *subcritical phase*.
(b) The system evolves to a steady state characterised by an evolving branching-annhilating pattern, we call this steady state the *supercritical phase*.

The separation between the two phases can be seen on Fig.1.b: the phase transition is materialised by the change in the average density (see legend). If the directed percolation hypothesis is valid, then theory and observations [11] predict that for an infinite lattice size system, the temporal evolution of the density $d_\alpha(t)$ obeys the following laws:

- for the critical value the density decreases as a power law: $d_{\alpha_c}(t) \sim t^{-\delta}$;
- for the supercritical phase $\alpha > \alpha_c$, the system converges to a stationary state characterised by a non-zero asymptotic density $d_\infty(\alpha)$. Near the critical point, the asymptotic density follows a power law: $d_\infty(\alpha) \sim (\alpha - \alpha_c)^\beta$.

We emphasize that the critical exponents $\delta = 0.1595$ and $\beta = 0.2765$ are known only experimentally (the values are given here with four digits, see [11] for better precision). They are valid for an initial random configuration where each cell has an equal probability to be in state 0 or 1.

Naturally, these predictions only hold for infinite systems ; as simulation requires finite lattices, we are bound to introduce finite-size effects. In the following section, we explain our protocol for measuring these exponents and limiting experimental errors.

3 Protocol

The measure of DP-critical exponents is a delicate operation that generally requires large amount of computation time. The main difficulty resides in avoiding systematic errors when obtaining statistical data near the transition point. For example, it happened that authors were mislead by their measures and concluded that a phase transition phenomenon was not in the DP universality class [13], which was later proved wrong [10].

In order to limit the influence of systematic errors, we use the two-step protocol that was used by Grassberger in [10]:

- We measure the critical synchrony rate α_c by varying α until we reach the best approximation of a power-law decay for the density. This first experiment also allows to measure the critical exponent δ.
- We measure the asymptotic density d_∞ as a function of α and then fit a power-law in order to calculate β.

Note that these two steps are not independent since the second operation uses the previously computed value of α_c.

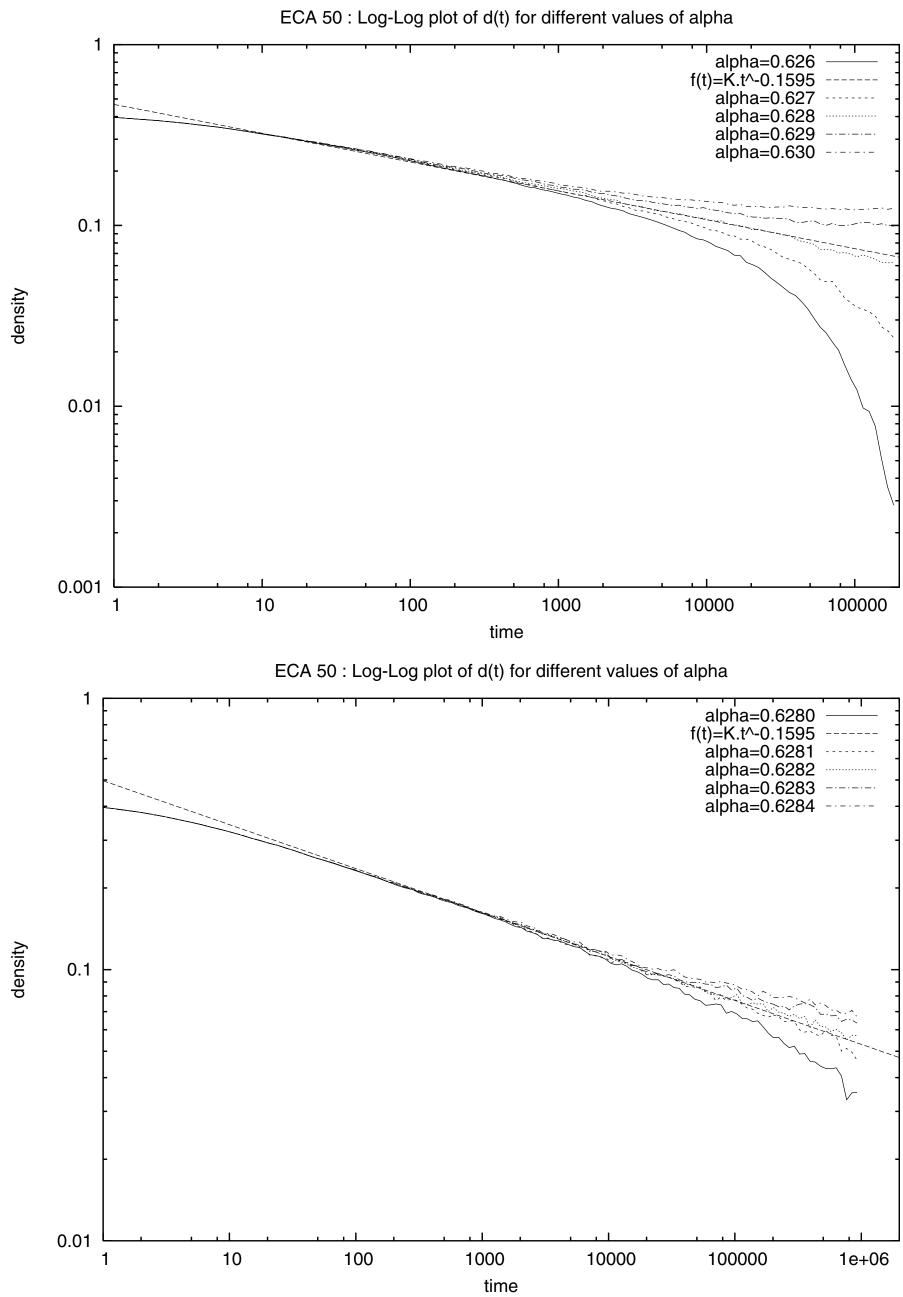

Fig. 2. ECA 50 : Determination of of the critical synchrony rate α_c using $d = f(t)$ for different values of α. Each curve shows the average obtained with on 100 runs. (a) α is varied by increments of 10^{-3}, for 2.10^5 simulation steps. (b) α is varied by increments of 10^{-4}, for 10^6 simulation steps. The straight line has slope $-\delta_{\text{DP}} = 0.1595$ and is plotted for reference.

In all the following experiments, we fixed the lattice size to $n = 10^4$ (as in [10]), verifying that the variations of the results when the system size was decreased to $n = 5000$ was less than the precision of measures. However, a more detailed protocol would require to do a *scaling analysis*, *i.e.*, to compute the limits of each measure as n goes to infinity.

3.1 Determination of δ

Figure 3 shows the temporal decay of the density for ECA 50 as α is varied by 10^{-3} steps from 0.626 to 0.630. The curves are obtained by averaging the data on 100 runs of time $T = 2.10^5$. We see that as α is increased, the curve in a log-log plot transforms from a concav function to a convex function ; the best linearity is obtained for $\alpha = 0.628$. As predicted, we see that the curve's slope in its linear part is close to $\delta_{DP} = 0.1595$.

In order to improve the precision on the measure of α_c, we repeated the previous experiment by varying α with a step of 10^{-4} using a sampling time $T = 10^6$. This operation is the most time-consuming as this computation requires more than 10^{14} applications of the local rule. The convexity of the curves was determined numerically, by plotting the local slope (see [11]) as a function of time according to:

$$\delta_{\mathrm{eff}}(t) = \frac{\log d(t) - \log d(t/m)}{\log t - \log(t/m)} = \frac{\log\left[d(t)/d(t/m)\right]}{\log m}$$

with m varying between 4 and 10 (heuristic criterion).

The values of α for which the best linearity was obtained are displayed in Table 1. The value of the slope, δ, is given in the third column of the table for comparison with δ_{DP}. We took $t \in [2000, 200000]$ as a fit interval to limit the influence of the transient time and the deviation from a power-law decay.

We wish to call the reader attention on the fact that we cannot identify the precision of this fit as the precision on the measure on δ. Indeed, as we are necessarily slightly subcritical or supecritical, the curve $d(t)$ eventually deviates from a power-law. To get an estimation on the precision on δ, we used the following heuristic method: if we bound the α_c according to $\alpha_1 < \alpha_c < \alpha_2$, we use the quantity $E_\delta = |\delta(\alpha_1) - \delta(\alpha_2)|$ as an estimator of the precision on δ. The results displayed in Table 1 show that the computed values of δ and E_δ are compatible with the predicted value δ_{DP}. It is interesting to notice that a variation on α of the order of 10^{-4} produces a relative variation of 10% on the value of δ. This explains why α_c has to be measured with high precision.

3.2 Determination of β

The second part of the experiments consists in measuring the critical exponent β using the values of the asymptotic density as a function of α. To estimate this asymptotic density, it is necessary to adjust the sampling time as α varies. Indeed, as α approaches α_c, the asymptotic density vanishes as :

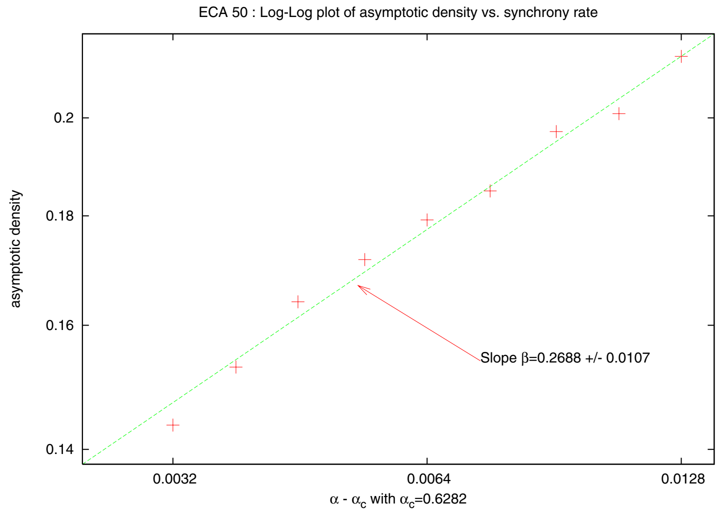

Fig. 3. ECA 50: Determination of the critical exponent β using the time decay properties. Each point is obtained according to using a particular sampling time (see text). The curve have slope $\beta_{50} = 0.2688 \pm 0.0107$. Note that both x and y axis are displayed in logarithmic scale.

$d_\infty(\alpha) = (\alpha - \alpha_c)^\beta$, and the increase of the time needed to reach this density is thus exponential: this phenomenon, known as *critical slowing down* (*e.g.*, [11]), limits the precision on the measure of the asymptotic density d_∞.

The quantity $\Delta_\alpha = \alpha - \alpha_c$ was varied according to an exponential increment from 0.0032 to 0.128. This interval is determined by the following trade-off: the computer time limits lower values of Δ_α (critical slowing down) and for higher values of Δ_α the system "saturates" and no longer follow a power-law. The deviation from the power law is a phenomenon that is predicted by theory and that can be studied for its own interest. However, we prefer here to restrict our measures to the linear part of the curve.

Sampling times were increased as Δ_α was decreased and the highest sampling time $T = 4.10^5$ was used for $\Delta_\alpha = 0.0032$. The experiment was conducted for the seven ECA and the calculated values are shown in Table 1. Again, the computed values of β are in good agreement with the reference value $\beta_{DP} = 0.2765$.

4 Discussion

The problem of determining how changes of behaviour were triggered by gradual changes in the update rule were investigated by numerical simulations. The results show good evidence that the phenomenon observed for seven asynchronous elementary cellular automata is a second order phase transition which belongs to the directed percolation universality class.

Table 1. Critical values $\tilde{\alpha}_c$ for the seven ECA with DP ; the digit between parentheses is uncertainty (in 10^{-4} units). Corresponding value of δ is given with an approximation of the error on δ (see text). Critical exponent β calculated using the given value $\tilde{\alpha}_c$.

ECA	$\tilde{\alpha}_c$	$\delta(\tilde{\alpha}_c)$	E_δ	β
6	0.2824 (4)	0.158	0.014	0.265 ± 0.015
18	0.7139 (2)	0.155	0.028	0.271 ± 0.009
26	0.4748 (2)	0.164	0.032	0.264 ± 0.015
50	0.6282 (2)	0.159	0.024	0.269 ± 0.011
58	0.3400 (2)	0.162	0.022	0.270 ± 0.017
106	0.8144 (4)	0.155	0.023	0.273 ± 0.048
146	0.6751 (2)	0.163	0.027	0.259 ± 0.021

The observation of the synchronous behaviour of the seven ECA studied indicate that there is certainly no straightforward relation with the existing classifications. For example, ECA 6, 50 and 58 are "periodic" (or Wolfram class II) rules while ECA 18, 26, 106 and 146 are "chaotic" (or Wolfram class III) rules. This indicates that at criticality, cell-scale details of cellular automata become irrelevant while some global scale-free behaviour governs its evolution.

These result may also further confirm a famous conjecture by Janssen and Grassberger (see [11] for a short presentation) that states that a model should belong to the DP universality class if it satisfies the following criteria:

- uniqueness of absorbing state (the all-zero state in our case),
- the possibility to characterise the phase transition by a positive order parameter (the density in our case),
- the definition of dynamics by short-range process (true by definition of CA),
- and the absence of additional symmetries (e.g., state symmetry) or quenched randomness (true for all of the seven ECA considered).

This last condition is essential since ECA 178, which is a rule symmetric by operation of left/right and 0/1 exchanging, was also detected to have a phase transition but was not found into the DP universality class.

The most challenging question now consists in explaining why some ECA show phase transitions while other have a smooth behaviour. A possibility of investigation would be to examine how the dynamics of asynchronous CA can be mapped with other well-studied phenomena such as synchronisation of configurations [10] or Domany-Kinzel probabilistic CA [5]. However, such a reduction does not appear simple since ECA 6 has an "inversed" phase transition: the subcritical (frozen) state is reached by the *increase* of the synchrony rate. Another interesting problem is to find examples of such phase transitions in nature. For example, this mechanism could help explaining the trigger of the self-organisation phase in cellular societies [2,6].

Acknowledgement. The author expresses his acknowledgments to A. Ballier, H. Berry, M. Morvan and J.-B. Rouquier.

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