Phase transitions in cellular automata : From computer science to statistical physics & back.

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Computer science

Elementary Cellular Automata (ECA) are discrete dynamical systems.

They are constituted of cells arranged linearly, each cell can be in two states symbolized by 0 and 1. In the infinite case, the cells are located on the bi-infinite line **Z**. In the finite case considered here, we arrange them in a ring **Z**/n**Z**.

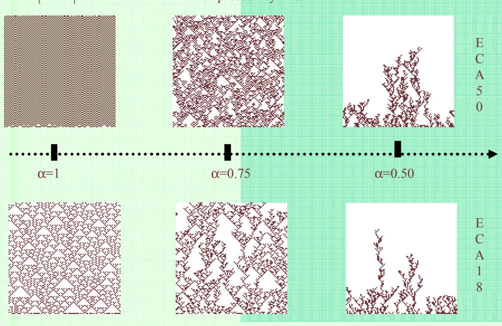
The state of the cell is represented by a configuration $x^t = (x_i^t) \in Q^{\wedge}\{\mathbf{Z}/n\mathbf{Z}\}$.

Classical ECA: In the classical synchronous regime, at each time step *t*, each cell *i* is updated according to a local function $f : \{0,1\}^3 \rightarrow \{0,1\}$.

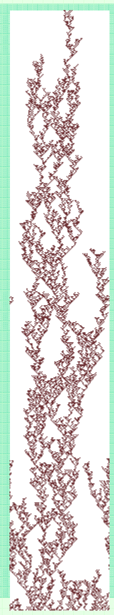
Asynchronous ECA: In the asynchronous regime, at each time step *t*, each cell *i* has a probability α to be updated : for all $i \in \mathbf{Z}/n\mathbf{Z}$,

$x_i^{t+1} = f(x_{i-1}^t, x_i^t, x_{i+1}^t)$ with probability α

$x_i^{t+1} = x_i^t$ with probability 1-α



In the experimental work [1], we showed that some ECA are "robust" to the change of synchrony rate α while other ECA displayed a sudden change of behaviour. Seven such ECA were identified ; **how to explain their change of dynamics ?**



Evolution of ECA 50 with α=60. Time goes from bottom to top.

Statistical physics

The percolation phenomenon is a well-studied model that shows phase transitions. In this model, sites are arranged on a diagonal square lattice. A link between two sites can be in two states : **open (blocking)** or **closed (porous)**.

In **isotropic percolation**, the links between two neighbouring sites are **closed** with probability *p* and **open** with probability *1-p*. A *cluster* is a maximal set of connected sites.

From a given site, what is the probability G(p) to obtain an "infinite cluster" ? There exists a critical probability p_c= ½, such that : *G(p)=0* for $p < p_c$ (dry phase), and *G(p)>0* for $p > p_c$ (wet phase).

Directed percolation is an anisotropic variant of percolation where the links between two sites are oriented according to a particular direction. It can be interpreted as a dynamical process where and the x-axis is space and the y-axis represents time.

The size of the clusters also diverge for a critical probability p_c . However, $p_c[DP] > p_c[IP]$.

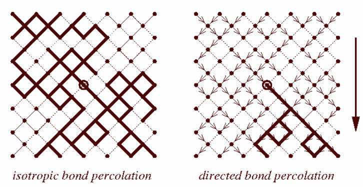


Illustration of percolation phenomena : courtesy of H. Hinrichsen [2]

Around criticality, experiments and theory predict that the density d(t) (average number of wet sites) evolves according to *p*:

For the critical phase $p = p_c$, the decrease follows : $d(t) \sim t^{\delta}$ where $\delta = -0.1595$… is known experimentally.

For the subcritical phase $p < p_c$, the density vanishes more rapidly $d(t) < t^{\delta}$.

For the supercritical phase $p > p_c$, the density stabilizes to an asymptotic value $d_{as}(p)$.

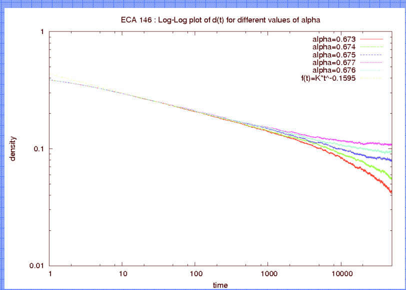
(2) The asymptotic density $d_{as}(p)$ diverges around p_c as :

$d_{as}(p) \sim (p - p_c)^{\beta}$ where $\beta = -0.2765$… is known experimentally.

The exponents **β** and **δ** define a **universality class** : many different phenomena obeying different microscopic rules exhibit power-laws with the same exponents. What about asynchronous ECA ?

Is the change of behaviour of asynchronous ECA in DP universality class ?

Finding the critical synchrony rate α_c



Each curve on this plot is obtained by computing the average of d(t) for ECA 146, on Z=100 runs, with a ring of size N=10 000, for a sampling time T=50 000.

The curve obtained for α=0.675 is almost straight and its slope coincides with the expected slope δ=-0.1595 (dashed line).

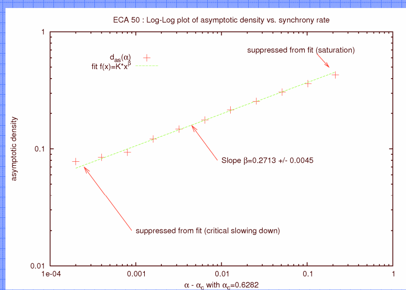
Repeating this experiment for the seven ECA, we obtain :

ECA	6	18	26	50	58	106	146
α_C	0.282	0.713	0.475	0.628	0.340	0.810	0.675

Limits of the protocol

Precision of α_c is less than 10^{-3}, the increase of precision implies more averaging and longer runs.

Finding the critical exponent β



Using the experimental value α_c, each point on this plot is obtained by computing the asymptotic density $d_{as}(\alpha - \alpha_c)$ for ECA 50. The size of the ring, the transient time, the sampling time are adapted for each measurement.

The aspect of the curve is linear in a log-log plot and the slope β is "almost" in agreement with the experimentally known value $\beta_{DP=}$ 0.2765.

Repeating this experiment for the seven ECA, we also obtain good agreement with directed percolation predictions.

Limits of the protocol

The values in left part of the curve are overestimated because of **the critical slowing down** phenomenon : the time needed to reach the asymptotic density grows exponentially as we approach criticality.

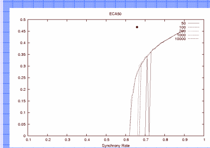
The right part of the curve shows **saturation** as the density can not exceed a certain limit.

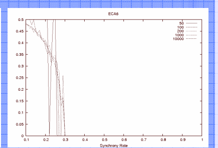
Discussion & openings

Computer science

Seven ECA that have different behaviours in synchronous mode turn out to have the same behaviour in some parts of their asynchronous regime : **asynchronism unveils complexity !**

The brutal change of attractor observed when changing α can be explained by the use of techniques from statistical physics. Experiments confirm that **there exists a phase transition** and that this phase transition is in the directed percolation universality class.





Linking the two models is still an open problem : how can we relate α and *p* ? Note that for ECA 6, it is the increase of alpha that leads to the sub-critical phase.

Statistical physics

The values of the critical exponents β and δ are only known by means of numerical simulations. Determining if they are rational numbers is still an open problem. This question is, according to P. Grassberger, one of the most challenging problems in the **study of phase transitions**. Is it possible to determine them from the analytical study of asynchronous CA ?

[1] N. Fatès and M. Morvan, An Experimental Study of Robustness to Asynchronism for Elementary Cellular Automata, *Complex Systems*, Volume 16, 2005.
[2] N.Fatès, *Robustesse de la dynamique des systèmes discrets : le cas de l'asynchronisme dans les automates cellulaires*, ENS Lyon thesis N° 04ENSL0298, 2004.
[3] H. Hinrichsen, Nonequilibrium Critical Phenomena and Phase Transitions into Absorbing States, *Advances in Physics*, Vol. 49, 2000.

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